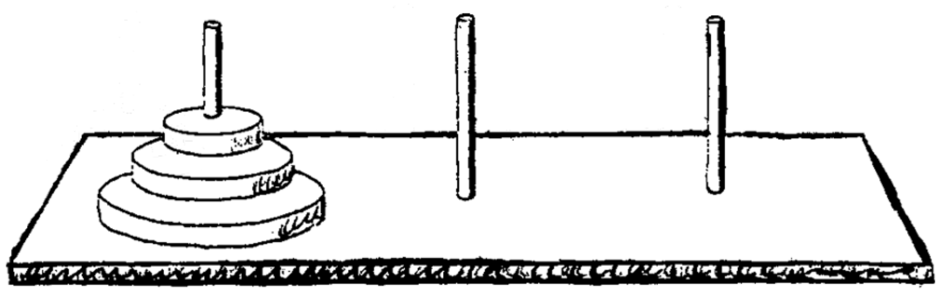
Tower of Hanoi

A legend about a group of ancient priests somewhere in the world tells of them solving a very old puzzle called the Tower of Hanoi. (You may be familiar with this puzzle from the computer program "Dr. Quandary®.")

The Tower of Hanoi is a set of 3 spindles or poles upon which are placed disks of descending sizes. (See graphic representation.)



The puzzle or challenge is to move the tower of disks to a different spindle following 2 rules:

- You may move only 1 disk at a time.
- At no time can a larger disk be placed on a smaller disk.

How many moves does it take to move the tower?

Last year's class made sets of these tower blocks having 5 blocks per set. How many moves does it take to move this tower?

What if you had a tower with 6 blocks? 7 blocks?

Extension for the Bold:

According to one version of the legend, the priests' puzzle has 64 gold disks on 3 diamond spindles. How many moves will it take them using the rules above?

Grade Levels 6 - 8

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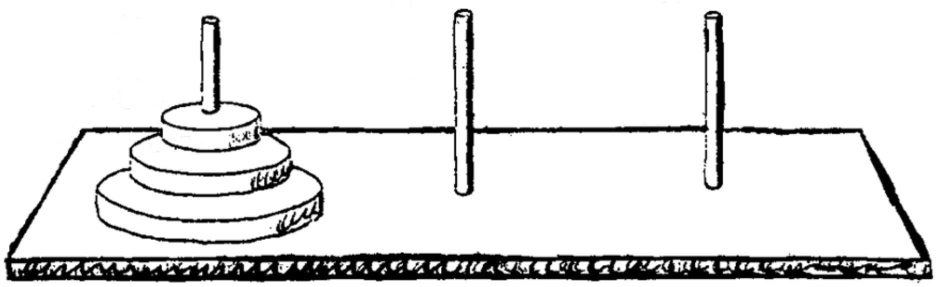

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Context

I was looking for a task that encouraged students to develop general rules from seeing patterns and this is such a task. Students had made several towers of wooden blocks in technology education and had been playing with them in the classroom before the task was given. There is also a segment in the computer program *Dr. Quandary®* that involves moving such a tower and the students had had experience with this also. There are tasks similar to this one in *Dale Seymour's Powers of Two Guide* and in *Equals Math for Girls and Other Problem Solvers*. You can purchase versions of this game at many puzzle and gift shops. You can also solve it using a collection of coins.

This task would be of most benefit with students who were familiar with the powers of two. Otherwise, transferring the pattern to a general rule may be impossible for them.

What This Task Accomplishes

By counting the number of moves needed to transfer towers of few blocks and keeping the results in an organized table, students should see a pattern based on powers of two and be able to form a general rule based on these observations. They can then use this rule to determine the number of moves for larger towers.

What the Student Will Do

By playing the game and recording the number of moves in an organized fashion, the student should see patterns develop. They should be able to transfer this information into a general rule to solve for towers too tall to physically move, while keeping track of the number of moves.

Time Required for Task

I suggest having the games out for several weeks in advance of giving the task so students have knowledge of how the game works and can concentrate on counting the moves.

The actual task can be done in one, 1 1/2 hours of class time with additional time outside of class for some students to complete the write-up.

Interdisciplinary Links

Students can construct these towers in technology education. My students made sets of five blocks from 1-1/2 inch to 6 1/2-inch square pieces of 3/4-inch pine boards.

Teaching Tips

Encouraging students to "solve a simpler problem" of moving a single block, a tower of two and a tower of three will help them to develop the pattern if they are having difficulty. Beware of the first student developing the general rule and spreading the information to others (see Apprentice Benchmark).

Suggested Materials

- *Towers of Hanoi
- Calculators - preferably scientific

*You could use blocks, commercially purchased or stacks of coins (dimes, pennies, nickels, quarters, half-dollars). Stacks of squares of colored paper in ascending sizes will also work. Students definitely need some manipulative materials to solve this task.

Possible Solutions

The general rule is 2n - 1 = number of moves needed to move the tower, with *n* being the number of blocks in the tower.

Benchmark Descriptors

Novice
This student does understand the basic premise of solving the tower problem for a tower of three blocks. What happens when s/he switches to coins? There is no record of results gathered. The general rule appears and had s/he tested it against his/her clear results for a tower of three blocks, s/he would have seen that it did not work. S/he says that *n* = the number of rings, but then uses the number of moves for *n* in his/her example. The graph is extremely unclear and only confuses the reader. This child did not know what s/he was doing after the original tower of five worked and never tried to verify his/her work to see his/her errors.

Apprentice
My suspicion is that this student overheard the general rule being discussed and copied it down, as there is no evidence for how s/he "found" the rule. The diagrams start out all right, but closer examination shows two blocks moved between step four and five. S/he did a good job of keeping track of the moves in a table, but one wonders how many of those numbers were actually discovered through the use of the manipulatives and at what point the general rule was applied. There is sparse mathematical language, even when there was the opportunity (ex. subtract instead of "take away"). There was no reason for this student to give up on finding the number of moves for a tower of 64, as the rule had been established and there were calculators available. This student took a minimalist approach to solving this problem.

Practitioner
This student did a good job of documenting his/her thinking throughout the task. While s/he never discovers the general rule, s/he has a clear understanding of the pattern s/he found and is able to use this pattern to solve all the towers required in the task. S/he skips from 16 to 32 and then to 64 on his/her chart, and I believe his/her answer for 64 blocks has a few too many zeros. S/he has good use of representations throughout the solution to document his/her work. It is easy for the reader to follow his/her thinking without his/her having to write volumes. This student is working on more efficient write ups. I think this is a good example of documenting work efficiently.

Expert

This student was able to discover a pattern and use the pattern until s/he could develop the general rule. S/he then used this general rule to solve for a tower of 64 blocks. This sixth grade student has not studied scientific notation enough to fully understand how to interpret the calculator's response to 264, but does explain to the reader how it was derived. The solution is clear, concise, shows good mathematical reasoning, uses good mathematical language and notation and adequate representation to inform the reader of the student's solution without being overly verbose. This was the best response I got to this task from my group of 23 sixth and seventh grade students.

Novice

The Tower of Hanoi

I started fooling around with the puzzle and figured it out I will show you the steps to figore it out.

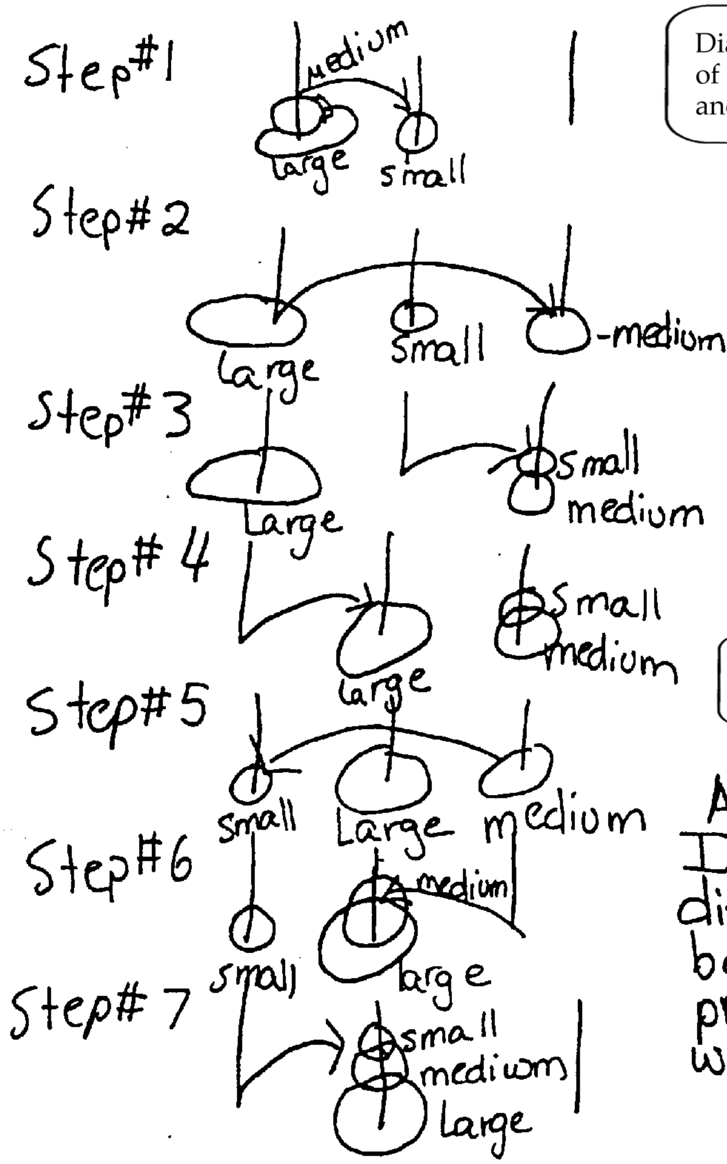


Diagram shows understanding of solving the original puzzle and counting moves.

It is unclear why using coins makes the problems easier.

After doing this I realized how difficult it will be with the other problems so I went to coins.

Novice

Using coins to figure the rest out was easier and more fun. It was also a time saver. I solved the other problems asked and found a general rule. It is as follows

n = # of rings on the spindles

n2 + 1

There is no record of the number of moves needed for a tower of five, six, seven or 64. Where did this rule come from?

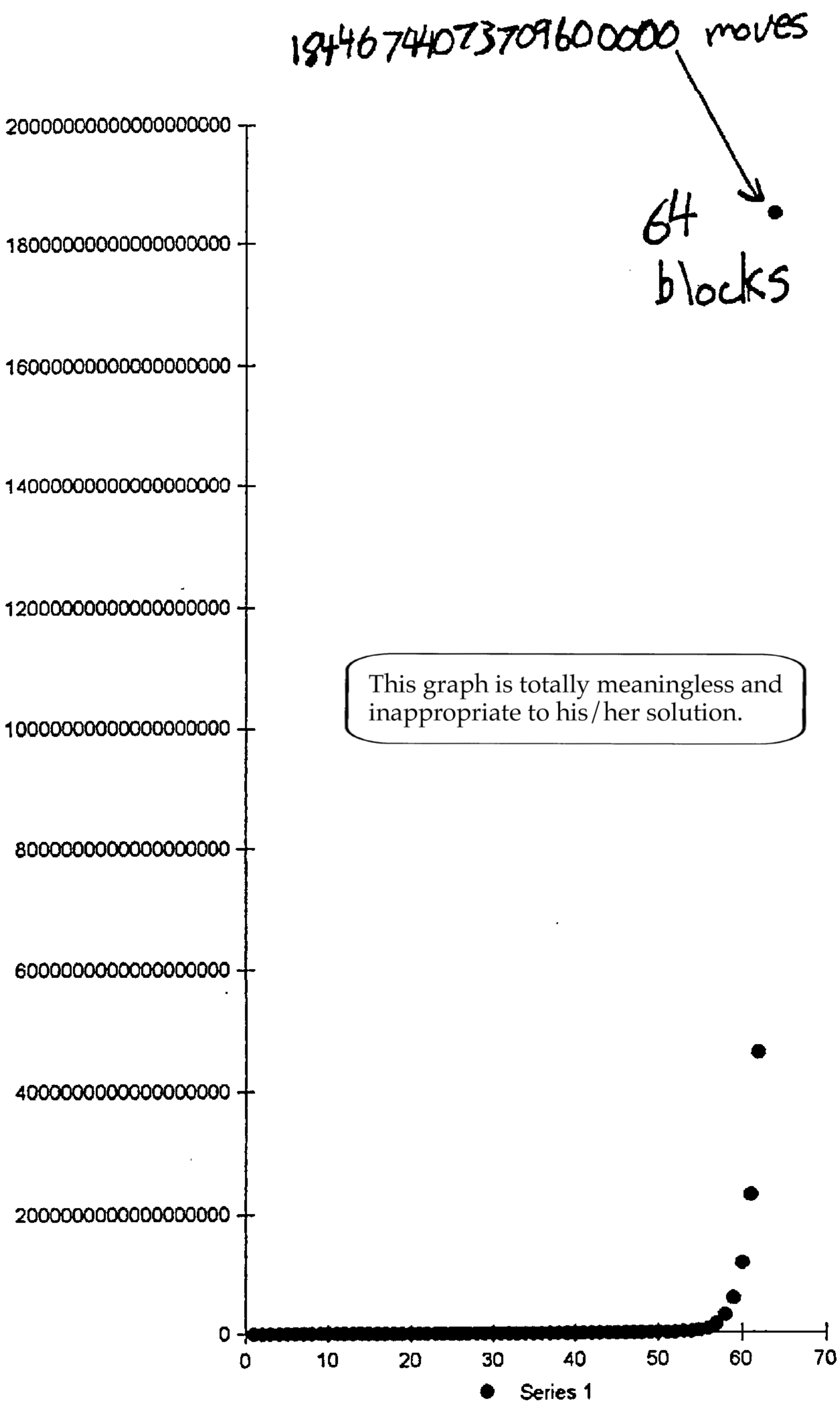
I will show you the other answers with the general rule.

How does s/he know that a tower of five takes 31 moves? Or, is 31 the number of rings?

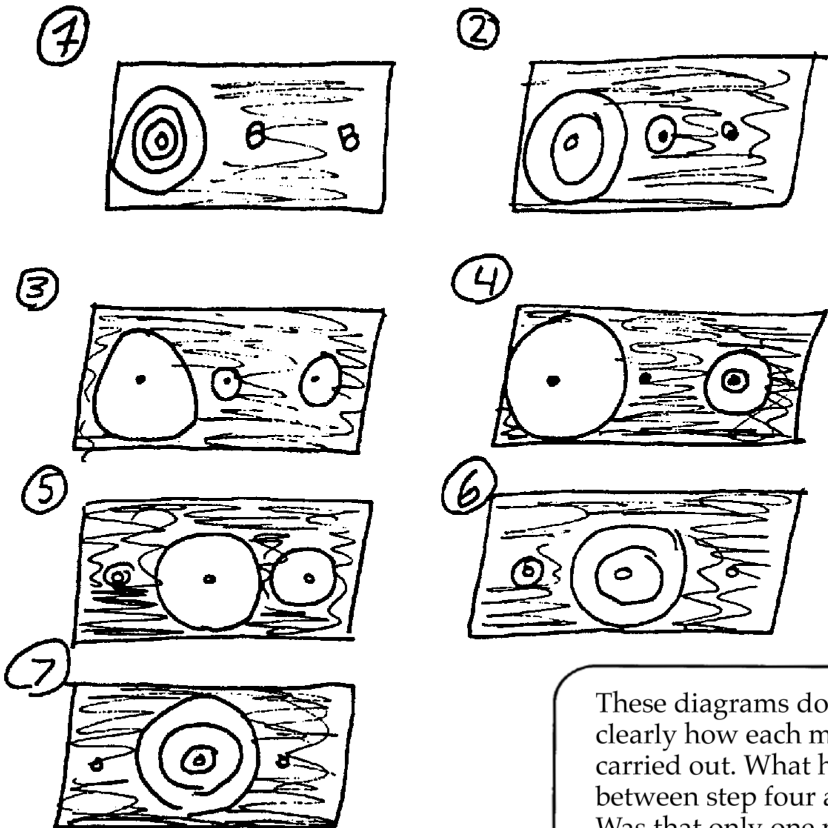
5 blocks per set 31 moves

$31 \cdot 2 + 1 = 63$

Novice


18446744073709600000 moves
64 blocks
This graph is totally meaningless and inappropriate to his/her solution.
20000000000000000000
18000000000000000000
16000000000000000000
14000000000000000000
12000000000000000000
10000000000000000000
8000000000000000000
6000000000000000000
4000000000000000000
2000000000000000000
0
0
10
20
30
40
50
60
70
Series 1

Apprentice



These diagrams do not show clearly how each move was carried out. What happened between step four and five? Was that only one move?

To do this problem I used munipulatives I discovered that to do it with three disks it takes 7 moves. I then found a general rule using the folowing information

# of disks	# of moves
1	1
2	3
3	7
4	15
5	31
6	63
7	128

Student gives no indication as to how the general rule was "found".

The general rule I found was: 2, 1.

Apprentice

This means that to solve for 64, I must do 2^{64} or 2 to the 64^{th} power, and than take away 1. This will give you your answer. I was unable to solve it though because the number is 32 digit number and is to hard to solve

The student should have used one of the scientific calculators available to solve this, or at least estimated the number of moves.

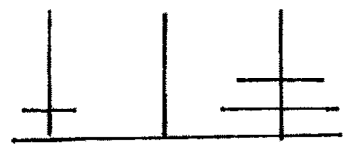
Practitioner

The problem: In this task I must move 3, 5, 6, and 7 disks from one pole to another without putting a larger on top of a smaller disk. I also must find out how many turns it takes. This is my step by step procedure.

How I did it:

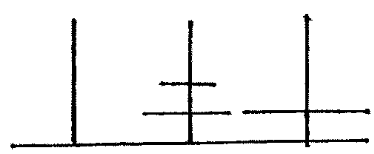
> Student shows clear understanding of solving the original task and explaining how s/he got the seven moves required.

I set up the board

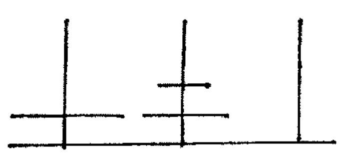


Move 1. I move the smallest block over to the last collum

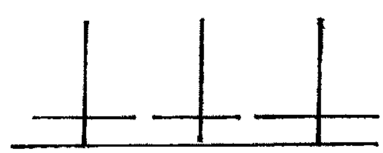
Move 2. I move the medium sized piec to the middle so next turn I can move the smaller piece on top.



Move3. I moved the small block on top of the medium block so I can move the big block next time.

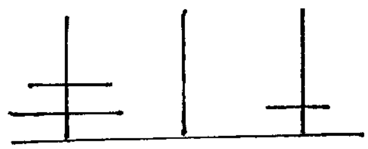


Move 4. I move the big big block because it is the base of the tower

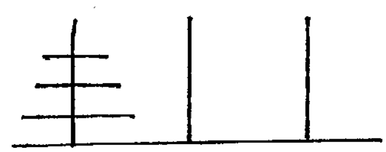


Move 5. I move the small block so that the medium block can go on top of the big block.

Practitioner



Move 6. I move the medium block on top of big block so that next turn I could put on the small block.



Move 7. I moved the small block to finish the tower.

I found out a pattern to get the answer for 5,6,& 7 blocks to get the answer easier. The # of moves for the previous # x 2 plus 1.

Blocks & Moves			
5	31	7	127
6	63		

> Recognizes that this is not a general rule, but rather is using the pattern to find the next solution.

So what: to extend this problem I will find out how many moves it takes to move a tower with 64 blocks. It takes 18,993,930,000,000,000,000,000,000 moves (It is a rounded #) My general rule (or pattern) is to get the # of moves of a D=double # of blocks you do: # of moves x (# of moves + 2).

Blocks	Moves	12	4,095
1	1	13	8,191
2	3	14	16,383
3	7	15	32,767
4	15	16	65,535
5	31	32	4,351,313,895
6	63	63	18,993,930,000,000,000,000,000,000, *
7	127		
8	255		
9	511		
10	1,023		
11	2,047		

Expert

The Tower Of Hanoi

In this task I was asked how many moves it would take to move a tower of three, a tower of six, a tower of seven, and a tower of 64 disks. I know that you can't put a larger disk on a smaller disk and that you can only move one disk at a time.

To solve this problem I first used three disks and counted my moves. I got seven. Then was the harder part. I had to make this table. To get the #s above five I had to use a pattern. It is: multiply the # of moves before it by two and then add one. I also found other patterns, but this is the one I used.

# of disks	Moves
1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255
9	511
10	1023
11	2047
12	4095

It would be interesting to figure out how long it would take to actually move the tower of 64 blocks.

Using this table I made a general rule for finding how many moves using the # of disks. It is: 2 to the x power (# of disks) -1= # of moves. So using this formula I tried 64 disks. I got 1.844674407 EE 19 (since my calculator only had ten digits this answer isn't down to the exact #). I could have tried the puzzle with 64 disks, but that would be counter productive and much to hard.

So, my answer is seven moves for three disks, 63 moves for six disks, 127 for seven disks, and 1.844674407 EE 19 for 64 disks.